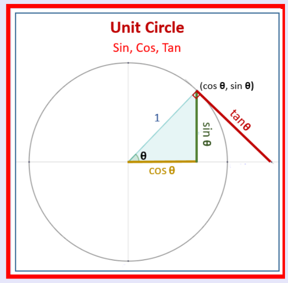
Math 241

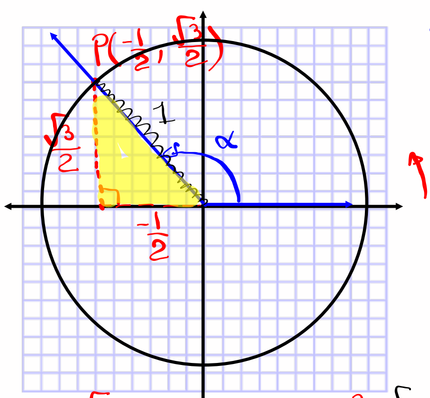
Winter 2023

Lecture 5



Consider the point $P\left(\frac{-1}{2}, \frac{\sqrt{3}}{2}\right)$ and angle $\boxed{\alpha > 0}$ such that its terminal side contains the point P.

1) Draw α is standard position.



α is in QII

2) Show that P is on the unit circle.

$x^2 + y^2 = 1$ ✓

$\left(\frac{-1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 =$

$\frac{1}{4} + \frac{3}{4} = \frac{4}{4} = 1$

3) $\sin\alpha = \frac{\sqrt{3}}{2}$ $\qquad$ $\csc\alpha = \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \boxed{\frac{2\sqrt{3}}{3}}$

$\cos\alpha = \frac{-1}{2}$ $\qquad$ $\sec\alpha = \frac{-2}{1} = \boxed{-2}$

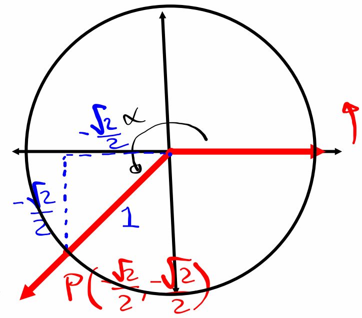
$\tan\alpha = -\sqrt{3}$ $\qquad$ $\cot\alpha = \frac{-1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \boxed{\frac{-\sqrt{3}}{3}}$

Given $P\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

1) Show P is on the Unit Circle.

$x^2+y^2=1 \checkmark$ $\quad \left(-\frac{\sqrt{2}}{2}\right)^2+\left(-\frac{\sqrt{2}}{2}\right)^2=\frac{2}{4}+\frac{2}{4}=\frac{4}{4}=1$

2) Draw angle $\boxed{\alpha>0}$ with terminal Side Contain the point P.



3) Find

$\sin\alpha=-\frac{\sqrt{2}}{2}$ $\qquad \csc\alpha=-\sqrt{2}$

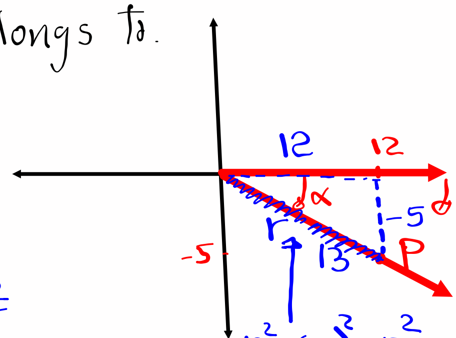
$\cos\alpha=-\frac{\sqrt{2}}{2}$ $\qquad \sec\alpha=-\sqrt{2}$

$\tan\alpha=+1$ $\qquad \cot\alpha=+1$

$$\frac{-2}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}=\frac{-2\sqrt{2}}{\sqrt{4}}=\frac{-\cancel{2}\sqrt{2}}{\cancel{2}}=-\sqrt{2}$$

Consider angle $\boxed{\alpha<0}$ Such that its terminal Side contain $P(12,-5)$.

1) Draw α in standard Position, and discuss which quadrant it belongs to.



2) Find

$\sin\alpha=-\frac{5}{13}$ $\qquad \csc\alpha=-\frac{13}{5}$

$\cos\alpha=+\frac{12}{13}$ $\qquad \sec\alpha=+\frac{13}{12}$

$\tan\alpha=-\frac{5}{12}$ $\qquad \cot\alpha=-\frac{12}{5}$

$12^2+(-5)^2=r^2$

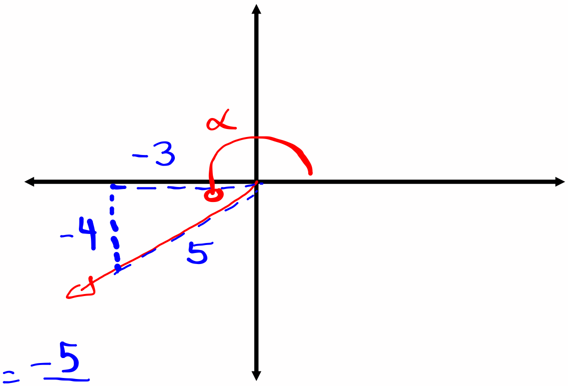
$144+25=r^2$

$169=r^2$

$\boxed{r=13}$

Given $\sin\alpha = \frac{-4}{5}$ and $\alpha > 0$ is in QIII.

1) Draw angle α.



2) Find

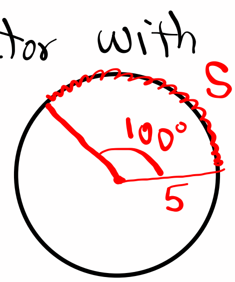
$\sin\alpha = \frac{-4}{5}$ $\qquad$ $\csc\alpha = \frac{-5}{4}$

$\cos\alpha = \frac{-3}{5}$ $\qquad$ $\sec\alpha = \frac{-5}{3}$

$\tan\alpha = +\frac{4}{3}$ $\qquad$ $\cot\alpha = +\frac{3}{4}$

Consider a circle with radius 5 ft.

1) Find the arc length of a sector with central angle of 100°.



$S = r\theta = 5 \cdot \frac{5\pi}{9} = \boxed{\frac{25\pi}{9}\text{ ft}}$

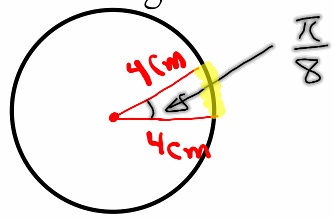
$180° = \pi$ $\qquad$ $1° = \frac{\pi}{180}$ $\qquad$ $100° = \frac{\cancel{100}^{5}\pi}{\cancel{180}_{9}}$

2) Find the area of the sector above.

$\text{Area} = \frac{1}{2}r^2\theta = \frac{1}{2}\cdot 5^2 \cdot \frac{5\pi}{9} = \boxed{\frac{125\pi}{18}\text{ ft}^2}$

A Sector has a central angle of $\frac{\pi}{8}$ with radius 4 cm.

1) Draw the Sector



2) Find the arc length of the Sector.

$$S = r\theta = 4 \cdot \frac{\pi}{8} = \boxed{\frac{\pi}{2} \text{ cm}}$$

3) Find its area.

$$\text{Area} = \frac{1}{2} r^2 \theta = \frac{1}{2} \cdot 4^2 \cdot \frac{\pi}{8} = \frac{16\pi}{16} = \boxed{\pi \text{ cm}^2}$$

Triangle ABC with $a=7$, $b=9$, and $c=10$.

1) Is ABC a right triangle?

$$a^2+b^2=c^2 \qquad 7^2+9^2=10^2 \qquad 49+81 \neq 100$$

NO

2) Find its area. Hint: Use Heron's Formula

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{a+b+c}{2}$

$$s = \frac{7+9+10}{2} = \frac{26}{2} = 13$$

$$\text{Area} = \sqrt{13(13-7)(13-9)(13-10)}$$

$$= \sqrt{13 \cdot 6 \cdot 4 \cdot 3}$$

$$= \sqrt{936} = 30.594$$

$$\boxed{\text{Area} \approx 31 \text{ units}^2}$$

Verify $\dfrac{1+\sin x}{1+\csc x} = \sin x$

$$\frac{1+\sin x}{1+\csc x} = \frac{1+\sin x}{1+\frac{1}{\sin x}} = \frac{\sin x(1+\sin x)}{\sin x\left(1+\frac{1}{\sin x}\right)}$$

multiply by $\sin x$

$$= \frac{\sin x(1+\sin x)}{\sin x + \cancel{\sin x}\cdot\frac{1}{\cancel{\sin x}}} = \frac{\sin x\,\cancel{(1+\sin x)}}{\cancel{\sin x + 1}}$$

$$= \boxed{\sin x}$$

Simplify $\dfrac{1+\sec x}{1+\cos x} = \dfrac{1+\frac{1}{\cos x}}{1+\cos x}$

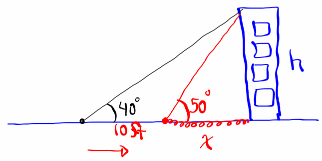
multiply by $\cos x$

$$= \frac{\cos x\left(1+\frac{1}{\cos x}\right)}{\cos x(1+\cos x)}$$

$$= \frac{\overset{1}{\cancel{\cos x + 1}}}{\cos x\,\cancel{(1+\cos x)}} = \frac{1}{\cos x}$$

$$= \boxed{\sec x}$$

Angle of elevation to the top of a building was 40°. If you walk 10 ft towards the building, angle of elevation becomes 50°. How tall is the building? Drawing Required.



$\tan 40° = \dfrac{h}{x+10}$

✓ $h = (x+10)\cdot \tan 40°$

$\tan 50° = \dfrac{h}{x}$

✓ $h = x \cdot \tan 50°$

$(x+10)\cdot \tan 40° = x \cdot \tan 50°$

$x \tan 40° + 10 \tan 40° = x \tan 50°$

$10 \tan 40° = x \tan 50° - x \tan 40°$

$10 \tan 40° = x\left[\tan 50° - \tan 40°\right]$

$x = \dfrac{10 \cdot \tan 40°}{\tan 50° - \tan 40°}$

$h = \dfrac{10 \cdot \tan 40°}{\tan 50° - \tan 40°} \cdot \tan 50° = \dfrac{10 \cdot \tan 40° \cdot \tan 50°}{\tan 50° - \tan 40°}$

$h \approx 28.356$ about 28 ft

Verify $\sin x(\sec x + \cot x) = \tan x + \cos x$ ✓

$\sin x(\sec x + \cot x) = \sin x\left(\dfrac{1}{\cos x} + \dfrac{\cos x}{\sin x}\right)$

$= \sin x \cdot \dfrac{1}{\cos x} + \cancel{\sin x} \cdot \dfrac{\cos x}{\cancel{\sin x}}$

$= \boxed{\dfrac{\sin x}{\cos x}} + \cos x$

$= \tan x + \cos x$ ✓

Verify: $\tan x - \cot x = \dfrac{\sin^2 x - \cos^2 x}{\sin x \cos x}$ ✓

$$\tan x - \cot x = \frac{\sin x}{\cos x} - \frac{\cos x}{\sin x}$$

$$= \frac{\sin x \cdot \sin x}{\cos x \cdot \sin x} - \frac{\cos x \cdot \cos x}{\sin x \cdot \cos x}$$

$$= \frac{\sin^2 x}{\cos x \sin x} - \frac{\cos^2 x}{\sin x \cos x} = \frac{\sin^2 x - \cos^2 x}{\sin x \cos x}$$

$$\frac{\sin^2 x - \cos^2 x}{\sin x \cos x} = \frac{\sin^{\cancel{2}} x}{\cancel{\sin x} \cos x} - \frac{\cos^{\cancel{2}} x}{\sin x \cancel{\cos x}}$$

$$= \frac{\sin x}{\cos x} - \frac{\cos x}{\sin x}$$

$$= \tan x - \cot x \checkmark$$

Verify $\dfrac{1}{1+\cos x} + \dfrac{1}{1-\cos x} = 2\csc^2 x$ ✓

$$\frac{1\,(1-\cos x)}{(1+\cos x)(1-\cos x)} + \frac{1\,(1+\cos x)}{(1-\cos x)(1+\cos x)} =$$

$$\frac{1 \cancel{-\cos x} + 1 \cancel{+\cos x}}{(1+\cos x)(1-\cos x)} = \frac{2}{1-\cos^2 x} =$$

$(A + B)(A - B)$

$A^2 - B^2$

$$= \frac{2}{\sin^2 x} = 2\csc^2 x$$

Verify $\dfrac{\tan^3\alpha - 1}{\tan\alpha - 1} = \sec^2\alpha + \tan\alpha$ ✓

Hint: $A^3 - B^3 = (A - B)(A^2 + AB + B^2)$

$$\frac{\tan^3\alpha - 1^3}{\tan\alpha - 1} = \frac{\cancel{(\tan\alpha - 1)}(\tan^2\alpha + \tan\alpha + 1)}{\cancel{\tan\alpha - 1}}$$

$$= \tan^2\alpha + \tan\alpha + 1$$

$1 + \tan^2\alpha = \sec^2\alpha$

$$= \boxed{\sec^2\alpha + \tan\alpha} ✓$$

Use $\alpha = 30^\circ$ to show $\sin\alpha\cos\alpha \neq 1$

$$\sin 30^\circ \cdot \cos 30^\circ = \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4} \neq 1.$$

Use $\theta = 45^\circ$ to show $\tan^2\theta + \cot^2\theta \neq 1$

$$\tan^2 45^\circ + \cot^2 45^\circ = 1^2 + 1^2 = 1 + 1 = 2 \neq 1$$

Verify

$$(\tan x + \cot x)^2 = \sec^2 x + \csc^2 x$$

Recall $(A+B)^2 = A^2 + 2AB + B^2$

$$(\tan x + \cot x)^2 = \tan^2 x + 2\underbrace{\tan x \cot x}_{1} + \cot^2 x$$

$$= \tan^2 x + 2 + \cot^2 x$$

$$= \underbrace{\tan^2 x + 1}_{} + \underbrace{1 + \cot^2 x}_{}$$

$$= \sec^2 x + \csc^2 x$$

Verify $\csc\theta + \sin(-\theta) = \dfrac{\cos^2\theta}{\sin\theta}$

Recall

$\sin(-\theta) = -\sin\theta$

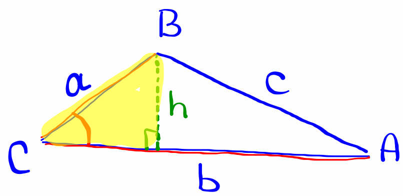
$$\csc\theta + \sin(-\theta) = \csc\theta - \sin\theta$$

$$= \frac{1}{\sin\theta} - \frac{\sin\theta \cdot \sin\theta}{\sin\theta}$$

$$= \frac{1}{\sin\theta} - \frac{\sin^2\theta}{\sin\theta} = \frac{1-\sin^2\theta}{\sin\theta}$$

$$= \frac{\cos^2\theta}{\sin\theta}$$

How to find area of a triangle when we have two sides and the angle between them:



$$\text{Area} = \frac{\text{base} \cdot \text{height}}{2} = \frac{b \cdot h}{2}$$

$$\sin C = \frac{h}{a} \Rightarrow h = a \sin C$$

$$\text{Area} = \frac{1}{2} b \cdot a \sin C$$

$$\text{Area} = \frac{1}{2} ab \sin C$$

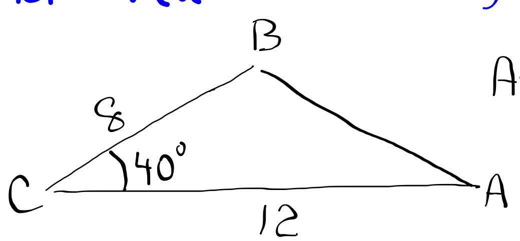
Similarly

$$\text{Area} = \frac{1}{2} ac \sin B$$

$$\text{Area} = \frac{1}{2} bc \sin A$$

Suppose $a = 8\text{cm}$, $b = 12\text{cm}$, $C = 40°$,

Find area of triangle ABC.



$$\text{Area} = \frac{1}{2} ab \sin C$$

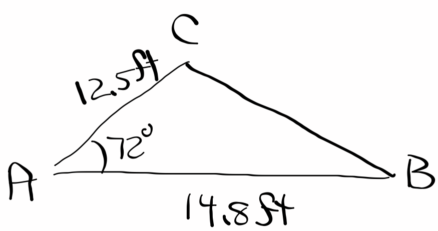
$$= \frac{1}{2} \cdot 8 \cdot 12 \cdot \sin 40°$$

$$= 48 \sin 40°$$

$$= 30.854$$

$$\approx 31 \text{ cm}^2$$

find area of a triangle with $b = 12.5$ ft, $C = 14.8$ ft and $A = 72°$.



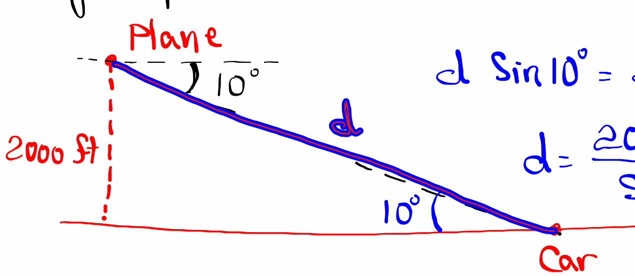
$$\text{Area} = \frac{1}{2} bc \sin A$$

$$= \frac{1}{2} \cdot (12.5)(14.8) \cdot \sin 72°$$

$$\text{Area} = \boxed{87.9}727\ldots$$

$$\boxed{\text{Area} \approx 88.0 \text{ ft}^2}$$

A plane is flying in altitude of 2000 ft. Pilot notices a parked car with angle of depression of 10°. How far is the plane from the car at the moment of viewing? Drawing Required.



$$\sin 10° = \frac{2000}{d}$$

$$d \sin 10° = 2000$$

$$d = \frac{2000}{\sin 10}$$

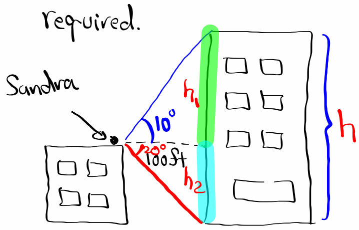
$d \approx 11517.541$

about 11,518 ft

Sandra is on the roof of a building 100 ft away from another building.

Her angle of elevation to the top of that building is 10°, and angle of depression to the bottom of the building is 20°.

How tall is the other building? Drawing required.



$\tan 10° = \frac{h_1}{100}$

$\boxed{h_1 = 100 \tan 10°}$

$\tan 20° = \frac{h_2}{100}$

$\boxed{h_2 = 100 \tan 20°}$

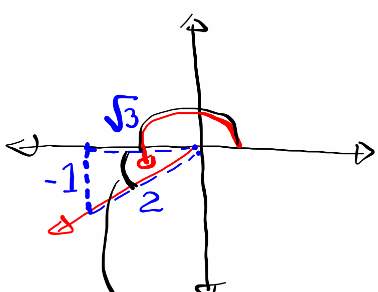
$h = h_1 + h_2$

$= 100 \tan 10° + 100 \tan 20°$

$= 100(\tan 10° + \tan 20°) \approx 54.030...$

about 54 ft tall

$\sin\theta = \frac{-1}{2}$, $\theta > 0$, θ is in QIII, Find θ.



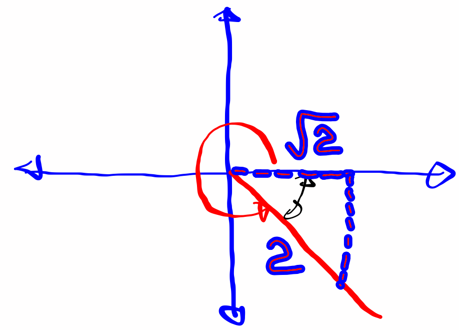
$\theta = 180° + 30°$

$\boxed{\theta = 210°}$

Ref. Angle $\Rightarrow$ $\sin ? = \frac{1}{2}$ R.A. $= 30°$

$\sec\theta = \sqrt{2}$, $\theta > 0$, θ is in QIV , Find θ

$\cos\theta = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$



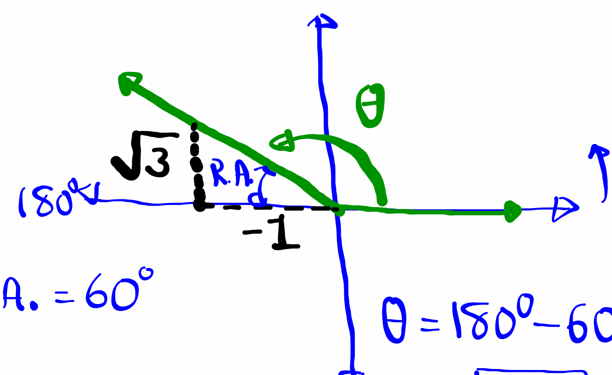
$\theta = 360^\circ - 45^\circ = \boxed{315^\circ}$

$\cos ? = \frac{\sqrt{2}}{2} \Rightarrow$ R.A. 45°

	30°	45°	60°
Sin			
Cos			
Tan			

$\tan\theta = -\sqrt{3}$, $\theta > 0$, θ is in QII , Find θ.

$-\sqrt{3} = \frac{-\sqrt{3}}{1}$

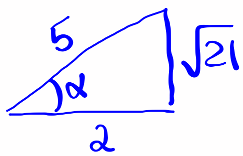


$\tan ? = \sqrt{3} \Rightarrow$ R.A. $= 60^\circ$

$\theta = 180^\circ - 60^\circ$

$= \boxed{120^\circ}$

$\cos\alpha = \frac{2}{5}$, α is in Q IV, find the remaining values of trig. functions.

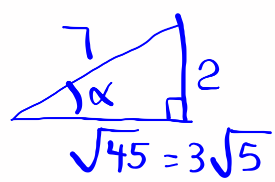


$\sin\alpha = -\frac{\sqrt{21}}{5}$ $\csc\alpha = -$ ✓

$\cos\alpha = \frac{2}{5}$ $\sec\alpha = \frac{5}{2}$

$\tan\alpha = -\frac{\sqrt{21}}{2}$ $\cot\alpha = -$ ✓

$\sin\alpha = \frac{2}{7}$, $\alpha > 0$, $\boxed{\alpha \text{ is in Q II}}$ Complete the chart below



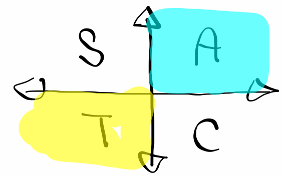
$\sin\alpha = \frac{2}{7}$ $\csc\alpha = \frac{7}{2}$

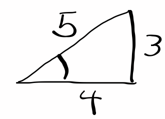
$\cos\alpha = -\frac{3\sqrt{5}}{7}$ $\sec\alpha = -$ ✓

$\tan\alpha = -\frac{2}{3\sqrt{5}} = -\frac{2\sqrt{5}}{15}$ $\cot\alpha = -$ ✓

$\tan\alpha = \frac{3}{4}$, $\alpha > 0$

find QIII





$\sin\alpha = \frac{-3}{5}$ $\quad$ $\csc\alpha = \frac{-5}{3}$

$\cos\alpha = \frac{-4}{5}$ $\quad$ $\sec\alpha = \frac{-5}{4}$

$\tan\alpha = \frac{3}{4}$ $\quad$ $\cot\alpha = \frac{4}{3}$

QI:

$\sin\alpha = \frac{3}{5}$ $\quad$ $\csc\alpha = \frac{5}{3}$

$\cos\alpha = \frac{4}{5}$ $\quad$ $\sec\alpha = \frac{5}{4}$

$\tan\alpha = \frac{3}{4}$ $\quad$ $\cot\alpha = \frac{4}{3}$

Law of Sines:

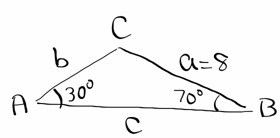
Any triangle $\quad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Ex: $A = 30°$, $B = 70°$, $a = 8\text{ cm}$

Solve $\underbrace{\text{the triangle}}$

find all missing sides and missing angles



$A + B + C = 180°$

$30° + 70° + C = 180°$

$\boxed{C = 80°}$

$$\frac{\sin 30°}{8} = \frac{\sin 70°}{b} = \frac{\sin 80°}{c}$$

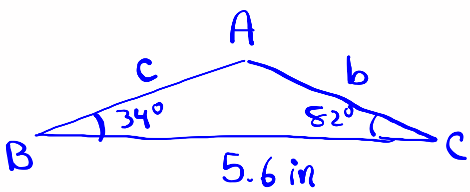
$$\frac{\sin 30°}{8} = \frac{\sin 70°}{b} \rightarrow b\sin 30° = 8\sin 70°$$

$$b = \frac{8\sin 70°}{\sin 30°} \quad \boxed{b \approx 15}\ \text{cm}$$

$$\frac{\sin 30°}{8} = \frac{\sin 80°}{c} \rightarrow c\cdot\sin 30° = 8\cdot\sin 80°$$

$$c = \frac{8\cdot\sin 80°}{\sin 30°} \quad \boxed{c \approx 16}\ \text{cm}$$

Solve triangle ABC if $B = 34°$, $C = 82°$, and $a = 5.6$ in.



$A + B + C = 180°$

$A + 34° + 82° = 180°$ $\boxed{A = 64°}$

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\boxed{\frac{\sin 64°}{5.6} = \frac{\sin 34°}{b}} = \frac{\sin 82°}{c}$$

$b \sin 64° = 5.6 \sin 34°$

$b = \dfrac{5.6 \sin 34°}{\sin 64°}$ $b \approx 3.5$ in.

Find c the same way.

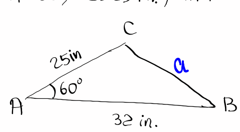
Law of Cosines:

Any triangle $a^2 = b^2 + c^2 - 2bc \cos A$

$b^2 = a^2 + c^2 - 2ac \cos B$

$c^2 = a^2 + b^2 - 2ab \cos C$

Find the missing parts of triangle ABC if $A = 60°$, $b = 25$ in., and $c = 32$ in.



$a^2 = b^2 + c^2 - 2bc \cos A$

$= 25^2 + 32^2 - 2(25)(32) \cdot \cos 60°$

$a^2 = 849$

$a = \sqrt{849}$ $\boxed{a \approx 29}$

$b^2 = a^2 + c^2 - 2ac \cos B$

$25^2 = 29^2 + 32^2 - 2(29)(32) \cos B$

$625 = 841 + 1024 - 1856 \cos B$

$1856 \cos B = 841 + 1024 - 625$

$1856 \cos B = 1240$ $\cos B = \dfrac{1240}{1856}$

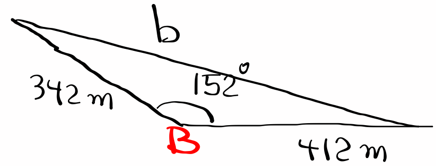
$\cos B = .668$ use $\cos^{-1}(.668) \approx \boxed{48°}$

More on this later

$A = 60°$
$B = 48°$ $\Rightarrow \boxed{C = }$

In triangle ABC, $a = 412\text{ m}$, $c = 342\text{ m}$, $B = 152^\circ$

Find side b.



Use Law of Cosines

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$= 412^2 + 342^2 - 2(412)(342)\cdot\cos 152^\circ$$

$$= 535529.6952$$

$$\boxed{b \approx 732\text{ m}}$$

$$b = \sqrt{535529.6952} = 731.798\ldots$$

Find one angle of triangle ABC such that

$a = 5$, $b = 8$, and $c = 12$.

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$2bc\cos A = b^2 + c^2 - a^2$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos A = \frac{8^2 + 12^2 - 5^2}{2(8)(12)}$$

$$\cos A = \frac{183}{192}$$

$$\cos A \approx .953$$

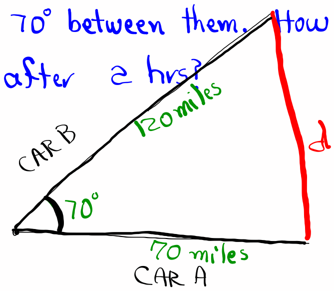
$$A = \cos^{-1}(.953)$$

$$\boxed{A \approx 18^\circ}$$

CAR A drives at 35 mph

CAR B " " 60 mph.

They leave at the Same on two roads with 70° between them. How far apart are they after 2 hrs?



d=r·t
=35·2
=70

d=r·t
=60·2
=120

$d^2 = 70^2 + 120^2 - 2(70)(120)\cdot \cos 70°$

$= 13554.06159$

$b = \sqrt{13554.06159}$

$b = 116.4219....$

about 116 miles apart